

# Spectral Evolution of the Universe

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## Abstract

We derive the evolution equations for the spectra of a space (“Universe”). Here the spectra mean the eigenvalues of the Laplacian on a space, which contain the geometrical information on the space.

These spectral evolution equations are expected to be useful to analyze the time evolution of the geometrical structures of the Universe. In particular, it is indispensable to investigate the time evolution of the spectral distance between two spaces, which is a measure of closeness between two geometries defined in terms of the spectra.

As an application, we investigate the time evolution of the spectral distance between two Universes that are very close to each other; it is the first necessary step for analyzing the time evolution of the geometrical discrepancies between the real Universe and its model. We find out a universal formula for the spectral distance between two very close Universes, which turns out to be independent of the detailed form of the spectral distance nor the gravity theory. Then we investigate its time evolution with the help of the spectral evolution equations. We also formulate the criteria for a good cosmological model in terms of the spectral distance.

## 1 Introduction

The notion of *closeness* or *distance* plays an essential role in physics. We frequently encounter the concept in the course of constructing and applying a the-

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ory.

The first step of finding out a law of nature is often classifying the objects in question into several categories (e.g., Hubble’s classification of galaxies). For classification, we are tacitly assuming the concept of closeness between the objects to be classified. Next, when we judge the validity of a theory by experiment, we construct a suitable parameter space associated to the theory and check the closeness between the point predicted by the theory and the experimental data-point in the parameter space. Here some kind of distance in the parameter space should be assumed. Finally, when we try to explain an observed phenomenon by a certain model based on a more or less established theory, we need to compare the observational data with the predictions of the model (e.g. the relation between a signal of the gravitational wave and its templates; a comparison of the observational data with the results of numerical simulations based on a model). Again some notion of distance is required in a parameter space suitable for this purpose.

Here we find out a universal setting for comparing “theory and reality”: There should be a parameter space equipped with the notion of closeness. We prepare a set of models (or templates). Each model corresponds to a point in the parameter space, so that the points corresponding to these models are distributed over the parameter space. Now, the observed data define another point in the parameter space. Then we try to find out the model-point that is closest to the data-point.

Thus we realize the significance of establishing a parameter space and a distance/closeness on it according to the problem we need to study.

The same situation occurs in cosmology and spacetime physics. Indeed, we often need to consider a set of Universes, rather than just only “our Universe”: Cosmology itself is a trial for grasping overall, averaged nature of the complicated reality of our Universe in terms of models; when we question why our Universe emerged rather than other possibilities, we are considering a set of Universes. According to the above considerations, thus, it is essential to establish a space of all Universes and a distance/closeness between any two Universes among them.

In a series of investigations, it turned out that we can in fact construct a space of all compact Universes equipped with a sort of distance.

Let *Riem* be the space of all  $(D - 1)$ -dimensional, compact Riemannian geometries without boundaries<sup>3</sup>. On *Riem*, we can introduce a measure of closeness in terms of the spectra, a set of eigenvalues of a certain elliptic operator [1]. Here we consider only the Laplacian  $\Delta$  as an elliptic operator. For a given geometry  $\mathcal{G} \in Riem$ , we get the *spectra*, or a set of eigenvalues of the Laplacian  $\{\lambda_n\}_{n=0}^{\infty} = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \uparrow \infty\}$ , numbered in an increasing order. Since  $\lambda_n$  has dimension  $[\text{Length}^{-2}]$ , the higher (lower) spectrum in general reflects the smaller (larger) scale properties of the geometry. Therefore the spectra

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<sup>3</sup> Throughout this paper,  $D$  represents the spacetime dimension, so that the spatial section is always  $(D - 1)$ -dimensional.

are desirable quantities for describing the effective geometrical structures of the space at each observational scale, e.g. the scale-dependent topology [2, 3]. Let us call this type of representation of geometry in terms of the spectra the *spectral representation* of geometrical structures [1].

Suppose  $\mathcal{G}$  and  $\mathcal{G}'$  are two spaces in *Riem*, and let  $\{\lambda_n\}_{n=0}^\infty$  and  $\{\lambda'_n\}_{n=0}^\infty$  be the spectra for  $\mathcal{G}$  and  $\mathcal{G}'$ , respectively. By comparing  $\{\lambda_n\}_{n=1}^N$  and  $\{\lambda'_n\}_{n=1}^N$  in a suitable manner, we can introduce a measure of closeness between  $\mathcal{G}$  and  $\mathcal{G}'$  of order  $N$  [1]. It compares two geometries up to the scale of order  $O(\lambda_N^{-1/2})$ , neglecting the smaller scale differences (we shall discuss more in detail on this topic in 4.2 and 4.3).

However, it turns out that this measure of closeness  $d_N(\mathcal{G}, \mathcal{G}')$  does not satisfy the triangle inequality though it satisfies the other two axioms of distance [1]. Even though the triangle inequality is far from a *must* from the viewpoint of the general theory of point set topology, it is certain that the inequality makes several arguments concise and it makes the measure of closeness more compatible with our notion of closeness.

This problem has been resolved by realizing that the breakdown of the triangle inequality is a mild one in a certain sense, and proving that *Riem* equipped with  $d_N(\mathcal{G}, \mathcal{G}')$  forms a metrizable space [4]. In other words, it has been justified to regard  $d_N(\mathcal{G}, \mathcal{G}')$  as a distance provided that a care is taken when the triangle inequality is required in the argument; we also found out the alternative of  $d_N(\mathcal{G}, \mathcal{G}')$  to be used when the triangle inequality is needed. (See 4.2 and Ref.[4] for more details.) As an immediate consequence, we can even introduce a distance on  $Slice(\mathcal{M}, g)$ , a space of all possible time-slices of a given spacetime  $(\mathcal{M}, g)$ . (See Ref.[7] for more details on this point and its application to the averaging problem in cosmology [8, 9].) From now on, we call  $d_N(\mathcal{G}, \mathcal{G}')$  the *spectral distance*.

Following the arguments at the beginning of this section, we have established a parameter space and a distance on it appropriate for spacetime physics: The space of all spaces of order  $N$ ,  $\mathcal{S}_N$ , which is a completion of  $(Riem, d_N)/\sim$  is what we had been searching for. (Here  $/\sim$  indicates the identification of isospectral manifolds [5]. We discuss a physical interpretation of the isospectral manifolds in Section 5. See also Ref.[4, 6].) Because of its nice property,  $\mathcal{S}_N$  can be regarded as a basic arena for the study of spacetime physics. For instance, we can define integral over  $\mathcal{S}_N$  [4], which would be essential in quantum cosmology.

Being  $\mathcal{S}_N$  at hand, we are now in a position to handle a set of Universes. For definiteness, let us focus on cosmological problems now. In cosmology, we need to judge to what extent a model reflects the real Universe. There is no guarantee whether cosmology is possible, viz. whether a model close to the reality at some instant of time, remains so all the time. From the viewpoint of the spectral representation, this fundamental problem can be visualized as follows: Let  $\mathcal{G}$  be the real Universe at present with respect to a certain time-slicing. Let  $\mathcal{G}'$  be a

model located in the neighborhood of  $\mathcal{G}$  in  $\mathcal{S}_N$ <sup>4</sup>. Then we should investigate the time evolution of  $d_N(\mathcal{G}, \mathcal{G}')$  and should analyze in what conditions  $d_N(\mathcal{G}, \mathcal{G}')$  remains small during a certain period of time.

This kind of investigation is now possible since the spectral distance is defined explicitly in terms of the spectra, which have a firm basis both physically and mathematically. What we now need is, thus, to analyze the time evolution of the spectra. In the spectral representation, the spectra  $\{\lambda_n\}$  are placed in the most fundamental position. Thus, from the purely theoretical viewpoint, too, it is interesting to investigate the time evolution of the spectra in detail.

As a first step, we can understand how the time evolution of the spectra is induced by the evolution of geometry as follows: By evolving an initial  $(D - 1)$ -dimensional geometry  $(\Sigma, h)$  according to the Einstein equations (in the Hamiltonian form if necessary), we get a 1-parameter family of geometries  $(\Sigma, h(t))$ . In principle, then, we can get the spectra for each geometry  $(\Sigma, h(t))$ . In this manner, we get a 1-parameter family of sets of spectra  $\{\lambda_n(t)\}$ .

It is more preferable both theoretically and practically, however, if the time evolution of  $\{\lambda_n(t)\}$  is described (1) solely in terms of spectral quantities, without any explicit reference to the metrical information behind them, and (2) in the form of differential equations of  $\{\lambda_n(t)\}$  with respect to time.

The key procedure for achieving this goal is to investigate the response of  $\{\lambda_n\}$  to the change of the spatial metric  $h$ . Since the latter is controlled by the Einstein equations, we thus expect to obtain the spectral version of the Einstein equations. The main aim of this paper is to obtain the fundamental evolution equations for the spectra, by putting this program into practice.

In section 2, we prepare several formulas that are needed in the subsequent investigations. In section 3, which is the main part of this paper, we derive the spectral evolution equations. As basic applications of the results we obtained, we discuss three topics in section 4. In 4.1, we study the spectral evolution of the closed Friedmann-Robertson-Walker Universe. In 4.2, we study the spectral distance between two Universes that are very close to each other in  $\mathcal{S}_N$ . We find out its universal expression in the leading order, which is independent of the detailed form of the spectral distance nor the gravity theory. In 4.3, we investigate the time evolution of the spectral distance between two very close Universes. Section 5 is devoted for discussions.

## 2 Basic formulas for the spectra

Let  $(\Sigma, g)$  be a  $(D - 1)$ -dimensional compact Riemannian manifold without boundaries. We set an eigenvalue problem for the Laplacian  $\Delta$  on  $(\mathcal{M}, g)$ ,

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<sup>4</sup>Here a model means a spacetime (usually it possesses much simpler geometrical structures than reality) along with a certain fixed time-slicing. We here regard an identical spacetime with different time-slicings as two different models.

$\Delta f = -\lambda f$ . Let  $\{\lambda_n\}_{n=0,1,2,\dots} := \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \uparrow \infty\}$  be the set of eigenvalues, or *spectra* hereafter, arranged in an increasing order. For simplicity of formulas, we assume that there is no degeneracy in the spectra throughout this paper. Let  $\{f_n\}_{n=0,1,2,\dots}$  be the set of real-valued eigenfunctions that are normalized as

$$(f_m, f_n) := \int_{\Sigma} f_m f_n \sqrt{g} = \delta_{mn} \quad , \quad (1)$$

where the natural integral measure on  $(\Sigma, g)$  is implied by  $\sqrt{g} := \sqrt{\det(g_{ab})}$ . Let us note that a set  $\{\frac{1}{\sqrt{\lambda_n}} \partial_a f_n\}_{n=1}^{\infty}$  forms an orthonormal subset of 1-forms on  $(\Sigma, g)$ ,

$$\frac{1}{\sqrt{\lambda_m \lambda_n}} \int_{\Sigma} \partial_a f_m g^{ab} \partial_b f_n \sqrt{g} = \delta_{mn} \quad . \quad (2)$$

## 2.1 Spectral components of functions and tensors

Let  $A$  and  $A_{ab}$  be any function and any symmetric tensor field, respectively, on  $(\Sigma, g)$ . It is useful to introduce diffeomorphism invariant quantities  $\langle A \rangle_{mn}$  and  $\langle A_{ab} \rangle_{mn}$  defined as

$$\begin{aligned} \langle A \rangle_{mn} &:= \int_{\Sigma} f_m A f_n \sqrt{g} \quad , \\ \langle A_{ab} \rangle_{mn} &:= \frac{1}{\sqrt{\lambda_m \lambda_n}} \int_{\Sigma} \partial^a f_m A_{ab} \partial^b f_n \sqrt{g} \quad . \end{aligned}$$

Here, for a quantity of type  $\langle A_{ab} \rangle_{mn}$ , we always understand that  $n, m \geq 1$  unless otherwise stated. We note that Eqs.(1) and (2) can be expressed as

$$\langle 1 \rangle_{mn} = \delta_{mn} \quad , \quad \langle g_{ab} \rangle_{mn} = \delta_{mn} \quad . \quad (3)$$

We also employ an abbreviated notation

$$\langle A \rangle_n := \langle A \rangle_{nn} \quad , \quad \langle A_{ab} \rangle_n := \langle A_{ab} \rangle_{nn} \quad .$$

For later uses, we develop these notations to

$$\begin{aligned} \langle A \rangle_{lmn} &:= \langle A f_l \rangle_{mn} \quad , \quad \langle A \rangle_{klmn} := \langle A f_k \rangle_{lmn} \quad , \dots \quad , \\ \langle A_{ab} \rangle_{l,mn} &:= \langle A_{ab} f_l \rangle_{mn} \quad , \quad \langle A_{ab} \rangle_{kl,mn} := \langle A_{ab} f_k \rangle_{l,mn} \quad , \dots \quad . \end{aligned}$$

We note that, for arbitrary functions  $A$  and  $B$

$$\langle AB \rangle_{mn} = \sum_{k=0}^{\infty} \langle A \rangle_{mk} \langle B \rangle_{kn} \quad . \quad (4)$$

To show this formula, we insert the  $\delta$ -function into the integral expression of  $\langle AB \rangle_{mn}$ , noting that  $\delta(x, y) \sqrt{g_y}^{-1} = \sum_{k=0}^{\infty} f_k(x) f_k(y)$ .

The following formula is also useful, which relates  $\langle Ag_{ab} \rangle_{mn}$  with  $\langle A \rangle_{mn}$ :

$$\langle Ag_{ab} \rangle_{mn} = \frac{\lambda_m + \lambda_n}{2\sqrt{\lambda_m \lambda_n}} \langle A \rangle_{mn} + \frac{1}{2\sqrt{\lambda_m \lambda_n}} \langle \Delta A \rangle_{mn} . \quad (5)$$

To show this formula, one modifies the defining equation for  $\langle Ag_{ab} \rangle_{mn}$  with the help of partial integrals, noting that the R.H.S. (right-hand side) should be symmetric in  $m$  and  $n$ .

Setting  $m = n$  in Eq.(5), we get

$$\langle Ag_{ab} \rangle_n = \langle A \rangle_n + \frac{1}{2\lambda_n} \langle \Delta A \rangle_n . \quad (6)$$

## 2.2 The variation formulas

We frequently consider the variation  $\delta Q$  of a certain quantity  $Q$  below. Here we treat  $\delta$  as a general variation for the time being. In later applications, the time-derivative (the Lie derivative along a time-flow vector  $t^a$  in a spacetime picture) is mostly considered as the variation operator  $\delta$ .

Now, noting that  $\Delta f = \frac{1}{\sqrt{}} \partial_a (\sqrt{ } g^{ab} \partial_b f)$  for an arbitrary function  $f$ , the variation of  $\Delta$  is represented as<sup>5</sup>

$$\delta \Delta f = \frac{1}{2} \partial_a (g \cdot \delta g) \partial^a f - \frac{1}{\sqrt{}} \partial_a (g^{ab} \delta g_{bc} \partial^c f \sqrt{ }) .$$

Thus, employing the same kind of notation as  $\langle A \rangle_{mn}$ , we can introduce the quantity

$$\langle \delta \Delta \rangle_{mn} := \int f_m \delta \Delta f_n \sqrt{ } = \frac{1}{2} \int f_m \partial_a (g \cdot \delta g) \partial^a f_n \sqrt{ } - \int f_m \partial_a (g^{ab} \delta g_{bc} \partial^c f_n \sqrt{ }) ,$$

where we note that the variation is taken only for the operator  $\Delta$ , and not for the eigenfunctions  $f_n, f_m$ . We should also keep in mind that  $\langle \delta \Delta \rangle_{mn}$  is *not* symmetric in  $m$  and  $n$ , unlike  $\langle A \rangle_{mn}$ , because  $\Delta$  is an operator, and not a function. Noting that  $f_0$  is a constant function (see *Appendix A*), it is evident that  $\langle \delta \Delta \rangle_{m0} = 0$  and  $\langle \delta \Delta \rangle_{0m} = \frac{\lambda_m}{2} \langle g \cdot \delta g \rangle_{0m}$  ( $m = 0, 1, 2, \dots$ ).

With these preliminaries, we now investigate the variations of spectral quantities. We start with the variation of the spectra. From Eq.(B10) in *Appendix B*, it follows that

$$\delta \lambda_n = -\langle \delta \Delta \rangle_n , \quad (7)$$

which is a basic result of the perturbation theory (“Fermi’s golden rule”).

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<sup>5</sup> To avoid trivial indices, we flexibly adopt notations such as  $\vec{u} := u^a$ ,  $A \cdot B := A_{ab} B^{ab}$ ,  $\vec{u} \cdot \vec{v} := u_a v^a$  and  $A \cdot := A_a^a$ . We also flexibly choose symbols for the derivative of a function  $f$ , such as  $D_a f = \vec{D} f = \partial_a f$ . (Here  $D_a$  and  $\vec{D}$  denote the covariant derivative.)

Now let us investigate the variation of the eigenfunctions. We have a general formula Eq.(B11) with Eq.(B12) (see *Appendix B*) for the perturbation of eigenvectors. Here we need to specify the factor  $c_n^{(1)}$  in Eq.(B12). For this purpose, we take the variation of the both sides of Eq.(1) for  $m = n$ . Noting that  $\delta\sqrt{ } = \frac{1}{2}g \cdot \delta g\sqrt{ }$ , we get

$$(f_n, \delta f_n) = -\frac{1}{4}\langle g \cdot \delta g \rangle_n . \quad (8)$$

In the standard perturbation analysis in quantum mechanics, the inner-product is fixed, and not perturbed, while the eigenfunctions are perturbed. Thus,  $\delta f_n$  should be perpendicular to  $f_n$  if  $f_n$  is normalized. In our case, on the other hand, the inner-product is also subject to the variation because of the presence of the integral measure  $\sqrt{ }$ . Thus  $\delta f_n$  is not in general perpendicular to  $f_n$ , as is clear in Eq.(8). Combining Eq.(8) with Eqs.(B11) and Eq.(B12), we see that  $c_n^{(1)}$  in Eq.(B12) should be chosen as  $c_n^{(1)} = -\frac{1}{4}\langle g \cdot \delta g \rangle_n$  in the present case. Thus, we get

$$\delta f_n = \sum_{k=0}^{\infty} f_k \mu_{kn} , \quad (9)$$

where

$$\mu_{mn} := \begin{cases} \frac{\langle \delta \Delta \rangle_{mn}}{\lambda_m - \lambda_n} & \text{for } m \neq n \\ -\frac{1}{4}\langle g \cdot \delta g \rangle_m & \text{for } m = n \end{cases} . \quad (10)$$

Taking the inner-product of the both sides of Eq.(9) with  $f_m$ , we get

$$\mu_{mn} = (f_m, \delta f_n) , \quad (11)$$

which gives a clear interpretation of the quantity  $\mu_{mn}$  as the projection of  $\delta f_n$  to the direction of  $f_m$ .

With the help of Eq.(9), it is now straightforward to show that

$$\delta \langle A \rangle_{mn} = \langle \delta A \rangle_{mn} + \frac{1}{2} \langle A g \cdot \delta g \rangle_{mn} + \sum_k \langle A \rangle_{mk} \mu_{kn} + \sum_k \langle A \rangle_{nk} \mu_{km} . \quad (12)$$

In particular, for the case of  $m = n$ , we get

$$\delta \langle A \rangle_n = \langle \delta A \rangle_n + \frac{1}{2} \langle A g \cdot \delta g \rangle_n + 2 \sum_k \langle A \rangle_{nk} \mu_{kn} . \quad (13)$$

Introducing  $\Gamma_{mn} := -\mu_{mn} - \frac{1}{4}\langle g \cdot \delta g \rangle_{mn}$  (note that  $\Gamma_{nn} = 0$ ), Eq.(12) is also represented as

$$\delta \langle A \rangle_{mn} = \langle \delta A \rangle_{mn} - \sum_k \langle A \rangle_{mk} \Gamma_{kn} - \sum_k \langle A \rangle_{nk} \Gamma_{km} .$$

It is interesting that this expression is in a similar form as the covariant derivative of a symmetric tensor.

In the same manner, we get a formula for  $\delta\langle A_{ab}\rangle_{mn}$  as

$$\begin{aligned}\delta\langle A_{ab}\rangle_{mn} &= \langle\delta A_{ab}\rangle_{mn} - (\delta\ln\sqrt{\lambda_m\lambda_n})\langle A_{ab}\rangle_{mn} + \frac{1}{2}\langle A_{ab} g \cdot \delta g\rangle_{mn} - \langle A_a^c \delta g_{cb} + A_b^c \delta g_{ca}\rangle_{mn} \\ &\quad + \sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle A_{ab}\rangle_{mk} \mu_{kn} + \sum_k \sqrt{\frac{\lambda_k}{\lambda_m}} \langle A_{ab}\rangle_{nk} \mu_{km} \ ,\end{aligned}\tag{14}$$

and for the case of  $m = n$ , we get

$$\begin{aligned}\delta\langle A_{ab}\rangle_n &= \langle\delta A_{ab}\rangle_n - (\delta\ln\lambda_n)\langle A_{ab}\rangle_n + \frac{1}{2}\langle A_{ab} g \cdot \delta g\rangle_n - 2\langle(A\delta g)_{ab}\rangle_n \\ &\quad + 2\sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle A_{ab}\rangle_{nk} \mu_{kn} \ .\end{aligned}\tag{15}$$

## 2.3 Basic identities

We obtain important relations by taking the variation of the basic identities Eq.(3).

First, we take the variation of the both sides of  $\delta_{mn} = \langle 1\rangle_{mn}$ . Then, with the help of Eq.(12), we get

$$0 = \frac{1}{2}\langle g \cdot \delta g\rangle_{mn} + \mu_{mn} + \mu_{nm} \ ,$$

which implies an identity,

$$\langle g \cdot \delta g\rangle_{mn} = -2(\mu_{mn} + \mu_{nm}) \ .\tag{16}$$

Thus, with the help of Eq.(10), we get

$$\langle g \cdot \delta g\rangle_{mn} = -\frac{2}{\lambda_m - \lambda_n}(\langle\delta\Delta\rangle_{mn} - \langle\delta\Delta\rangle_{nm}) \text{ for } m \neq n \ .\tag{17}$$

On the other hand, no condition is imposed on  $\langle g \cdot \delta g\rangle_n$  except for

$$\langle g \cdot \delta g\rangle_0 = 2\frac{\delta V}{V} \ ,\tag{18}$$

which follows by an independent argument (see *Appendix A*).

Now we take the variation of the both sides of  $\delta_{mn} = \langle g_{ab}\rangle_{mn}$  for  $m, n \geq 1$ . Then, with the help of Eq.(14), we get

$$0 = -\langle\overline{\delta g}_{ab}\rangle_{mn} - (\delta\ln\lambda_n)\delta_{mn} + \sqrt{\frac{\lambda_m}{\lambda_n}} \mu_{mn} + \sqrt{\frac{\lambda_n}{\lambda_m}} \mu_{nm} \ ,\tag{19}$$

where  $\overline{A}_{ab} := A_{ab} - \frac{1}{2}A \cdot g_{ab}$  for any symmetric tensor  $A_{ab}$ .



Thus, we get

$$\langle \overline{\delta g_{ab}} \rangle_{mn} = \begin{cases} \sqrt{\frac{\lambda_m}{\lambda_n}} \mu_{mn} + \sqrt{\frac{\lambda_n}{\lambda_m}} \mu_{nm} \\ = \frac{1}{\lambda_m - \lambda_n} \left( \sqrt{\frac{\lambda_m}{\lambda_n}} \langle \delta \Delta \rangle_{mn} - \sqrt{\frac{\lambda_n}{\lambda_m}} \langle \delta \Delta \rangle_{nm} \right) & \text{for } m \neq n \\ -\delta \ln \lambda_n - \frac{1}{2} \langle g \cdot \delta g \rangle_n & \text{for } m = n \end{cases} . \quad (20)$$

The fact that  $\langle \delta \Delta \rangle_{m0} = 0$  ( $m = 0, 1, 2, \dots$ ) suggests us to formally define that  $\langle \overline{\delta g_{ab}} \rangle_{m0} = 0$  ( $m = 0, 1, 2, \dots$ ). We adopt this formal definition here for a notational neatness (see Eq.(21) below).

From Eqs.(17) and (20), we obtain a formula for  $\langle \delta \Delta \rangle_{mn}$ ,

$$\langle \delta \Delta \rangle_{mn} = \frac{\lambda_n}{2} \langle g \cdot \delta g \rangle_{mn} + \sqrt{\lambda_m \lambda_n} \langle \overline{\delta g_{ab}} \rangle_{mn} . \quad (21)$$

We can also derive Eq.(21) directly from the definition of  $\langle \delta \Delta \rangle_{mn}$ .

We note that Eq.(21) is valid for the case of  $m = n$  also, due to Eq.(7) and the second equation of Eq.(20). Furthermore, we realize that this formula is also valid for  $m = 0$  or  $n = 0$  on account of the formal definition  $\langle \overline{\delta g_{ab}} \rangle_{m0} = 0$  ( $m = 0, 1, 2, \dots$ ).

We now investigate a different type of identities.

We pay attention to the second equation in Eq.(20),

$$\delta \ln \lambda_n = -\langle \overline{\delta g_{ab}} \rangle_n - \frac{1}{2} \langle g \cdot \delta g \rangle_n . \quad (22)$$

From this relation, it is straightforward to show a formula

$$\int A_{ab} \frac{\delta \ln \lambda_n}{\delta g_{ab}} = -\langle \overline{A_{ab}} \rangle_n - \frac{1}{2} \langle A \cdot \rangle_n , \quad (23)$$

where  $A_{ab}$  is any symmetric tensor field.

Let  $u^a$  be any vector field. Substituting  $A_{ab} = \mathcal{L}_{\vec{u}} g_{ab}$ , the L.H.S. (left-hand side) of Eq.(23) vanishes because of the diffeomorphism invariance of the spectra, so that

$$\begin{aligned} \langle \mathcal{L}_{\vec{u}} g_{ab} \rangle_n &= \langle \vec{D} \cdot \vec{u} g_{ab} \rangle_n - \langle \vec{D} \cdot \vec{u} \rangle_n , \\ \langle \overline{\mathcal{D}_{(a} u_{b)}} \rangle_n + \frac{1}{2} \langle \vec{D} \cdot \vec{u} \rangle_n &= 0 . \end{aligned} \quad (24)$$

We now note another identity. Using the basic properties of the covariant derivative  $D_a$ , it is easily shown that

$$\Delta D_a f = D_a \Delta f + \mathbf{R}_{ab} D^b f ,$$

where  $f$  is any smooth function. In particular, choosing  $f_n$  as  $f$ , we get

$$(g_{ab} \Delta - \mathbf{R}_{ab}) \partial^b f_n = -\lambda_n \partial_a f_n , \quad (25)$$

i.e.  $\partial_a f_n$  turns out to be an eigenfunction of  $g_{ab}\Delta - \mathbf{R}_{ab}$  with the eigenvalue  $\lambda_n$ . Taking the inner-product of Eq.(25) with  $\partial_a f_m$ , we get

$$\int D_a D_b f_m D^a D^b f_n \sqrt{\gamma} = -\sqrt{\lambda_m \lambda_n} \langle \mathbf{R}_{ab} \rangle_{mn} + \lambda_n^2 \delta_{mn} \quad , \quad (26)$$

or

$$\langle \mathbf{R}_{ab} \rangle_{mn} = \langle g_{ab} \Delta \rangle_{mn} + \lambda_n \delta_{mn} \quad . \quad (27)$$

### 3 Evolution equations for the spectral quantities

Now we investigate the time evolution of the spectra of a space. We have prepared in the previous section basic formulas regarding the responses of the spectral quantities with respect to a change  $\delta g_{ab}$ . When we let  $\delta g_{ab}$  be of a dynamical origin, thus, we automatically obtain the evolution equations for the spectral quantities.

We consider  $(\Sigma, h)$ , a  $(D-1)$ -dimensional compact Riemannian manifold without boundaries, as a mathematical model of the spatial section of the Universe. For the present purpose, we interpret a quantity  $\delta Q$  as  $\frac{d}{d\alpha}|_{\alpha=0} Q(\alpha)$ , and identify the latter quantity with  $\dot{Q} := \mathcal{L}_{\vec{t}} Q$ , where  $\vec{t}$  is a time-flow vector<sup>6</sup>. In particular, we replace  $\delta g_{ab}$ ,  $g \cdot \delta g$  and  $\overline{\delta g_{ab}}$  in the previous section with corresponding quantities as

$$\begin{aligned} \delta g_{ab} &\longmapsto \dot{h}_{ab} = 2NK_{ab} + 2D_{(a}N_{b)} \quad , \\ g \cdot \delta g &\longmapsto h^{ab}\dot{h}_{ab} = 2NK + 2\vec{D} \cdot \vec{N} \quad , \\ \overline{\delta g_{ab}} &\longmapsto \overline{\dot{h}_{ab}} = 2N\overline{K}_{ab} + 2\overline{D_{(a}N_{b)}} \quad . \end{aligned} \quad (28)$$

Here  $K_{ab}$  is the extrinsic curvature and  $K := K^a_a$ ;  $N$  and  $N_a$  are the lapse function and the shift vector, respectively.

#### 3.1 Evolution equation for $\{\lambda_n\}$

First, Eq.(22) becomes

$$\begin{aligned} \dot{\lambda}_n &= -\left(2\langle N\overline{K}_{ab} \rangle_n + \langle NK \rangle_n\right) \lambda_n \\ &= -2\langle NK_{ab} \rangle_n \lambda_n + \frac{1}{2}\langle \Delta(NK) \rangle_n \quad , \end{aligned} \quad (29)$$

where we used the identity Eq.(24) in the first line, and Eq.(6) in the second line. We note that the shift vector  $N_a$  does not appear in Eq.(29), reflecting the spatial diffeomorphism invariance of  $\lambda_n$ .

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<sup>6</sup> See the argument at the end of *Appendix B* also.

Now, Eqs.(16) and (20) becomes

$$\begin{aligned} \langle NK \rangle_{mn} + \langle \vec{D} \cdot \vec{N} \rangle_{mn} &= -(\mu_{mn} + \mu_{nm}) \\ \langle N \overline{K}_{ab} \rangle_{mn} + \frac{1}{2}(\ln \lambda_n) \delta_{mn} + \langle \overline{D}_{(a} \overline{N}_{b)} \rangle_{mn} &= \frac{1}{2} \left( \sqrt{\frac{\lambda_m}{\lambda_n}} \mu_{mn} + \sqrt{\frac{\lambda_n}{\lambda_m}} \mu_{nm} \right) , \end{aligned}$$

where  $\mu_{mn}$  is given by Eq.(10) with  $\langle \delta \Delta \rangle_{mn}$  replaced by  $\langle \dot{\Delta} \rangle_{mn}$ .

Eq.(21) becomes

$$\langle \dot{\Delta} \rangle_{mn} = \lambda_n \langle NK \rangle_{mn} + 2\sqrt{\lambda_m \lambda_n} \langle N \overline{K}_{ab} \rangle_{mn} + \lambda_n \langle \vec{D} \cdot \vec{N} \rangle_{mn} + 2\sqrt{\lambda_m \lambda_n} \langle \overline{D}_{(a} \overline{N}_{b)} \rangle_{mn} . \quad (30)$$

Ultimately we are only interested in the dynamics of the spectra  $\{\lambda_n\}$ , which is invariant under the spatial diffeomorphism. The coordinate dependence of other subsidiary variables like  $\mu_{mn}$ , which can dependent on the shift vector  $N_a$ , should not have any influence on the dynamics of  $\{\lambda_n\}$ . Hence we can set  $N_a = 0$  from the very beginning to make formulas simpler. For simplicity we also set  $N = 1$  hereafter.

Now let us investigate the evolution equations for the spectra, Eq.(29), in detail. At this stage, it is useful to develop notations. First, we note that we can expand any scalar function  $A(\cdot)$  in terms of  $\{f_n\}_{n=0}^\infty$ :

$$A(\cdot) = \sum_{l=0}^{\infty} A_l f_l(\cdot) . \quad (31)$$

Then, it follows that

$$\Delta A(\cdot) = - \sum_{l=1}^{\infty} \lambda_l A_l f_l(\cdot) ,$$

where we see that the homogeneous component of  $A$ ,  $A_0$ , does not appear in the summation.

The homogeneous component  $A_0$  is related to the spatial average of  $A$  over the spatial section  $\Sigma$ ,  $A_{\text{av}} := \frac{1}{V} \int_{\Sigma} A \sqrt{\gamma}$ , as

$$A_{\text{av}} = A_0 / \sqrt{V} . \quad (32)$$

Next, let us introduce the quantity

$$(l \ m \ n) := \langle f_m \rangle_{ln} = \langle 1 \rangle_{mln} = \int f_l f_m f_n \sqrt{\gamma} . \quad (33)$$

Note that  $(l \ m \ n)$  is totally symmetric in  $l$ ,  $m$  and  $n$ . We can also introduce a similar quantity  $(l \ m \ n \ k) := \int f_l f_m f_n f_k \sqrt{\gamma}$ , and similarly  $(l \ m \ n \ k \ h)$ , and so on. However, the quantities of the type  $(l \ m \ n)$  are sufficient since other quantities can be expressed in terms of  $(l \ m \ n)$ . For instance,

$$(l \ m \ n \ k) = \frac{1}{3} \sum_{l'} \{ (l \ m \ l')(l' \ n \ k) + (m \ n \ l')(l' \ k \ l) + (n \ k \ l')(l' \ l \ m) \} ,$$

due to Eq.(4).

We note that the quantity of the form  $\langle A \rangle_{lmn\dots}$  is represented in terms of  $A_l$  and  $(l\ m\ n)$  since

$$\langle A \rangle_{lmn\dots} = \sum_{l'} A_{l'}(l' \ l \ m \ n \ \dots) \ .$$

In the same manner, the quantities of the form  $\langle A_{ab} \rangle_{kl\dots,mn}$  is represented in terms of  $\langle A_{ab} \rangle_{l,mn}$  and  $(l\ m\ n)$  because of the relation

$$\langle A_{ab} \rangle_{kl\dots,mn} = \sum_{l'} \langle A_{ab} \rangle_{l',mn} (l' \ k \ l \ \dots) \ .$$

In particular, the quantity of the form  $\langle A_{ab} \rangle_{mn}$  can be represented in terms of  $\langle A_{ab} \rangle_{0,mn}$  as

$$\langle A_{ab} \rangle_{mn} = \langle A_{ab} \rangle_{0,mn} \sqrt{V} \ .$$

Keeping the applications to cosmology in mind, it is also useful to introduce the quantities  $\epsilon_{ab}$  and  $r_{ab}$  defined as

$$\begin{aligned} \epsilon_{ab} &:= K_{ab} - \frac{1}{D-1} K h_{ab} \ , \\ r_{ab} &:= \mathbf{R}_{ab} - \frac{1}{D-1} \mathbf{R} h_{ab} \ . \end{aligned} \quad (34)$$

Note that  $\epsilon_{\cdot\cdot} = r_{\cdot\cdot} = 0$ . The quantities  $\epsilon_{ab}$  and  $r_{ab}$  characterize the deviation of the spatial geometry  $(\Sigma, h)$  from the isotropic geometry.

Now, we note that

$$\langle K \rangle_{mn} = \sum_l K_l (l\ m\ n) \ ,$$

and

$$\begin{aligned} \langle \bar{K}_{ab} \rangle_{mn} &= -\frac{D-3}{2(D-1)} \langle K h_{ab} \rangle_{mn} + \langle \epsilon_{ab} \rangle_{mn} \\ &= -\frac{D-3}{4(D-1)} \frac{1}{\sqrt{\lambda_m \lambda_n}} \sum_l (\lambda_m + \lambda_n - \lambda_l) K_l (l\ m\ n) + \langle \epsilon_{ab} \rangle_{mn} \ . \end{aligned}$$

Here Eq.(31) has been applied to  $K$  and we have used Eq.(5).

Thus, Eq.(29) is represented as

$$\dot{\lambda}_n = -\frac{2}{D-1} \sum_l \left( \lambda_n + \frac{D-3}{4} \lambda_l \right) K_l (l\ n\ n) - 2\lambda_n \langle \epsilon_{ab} \rangle_n \ . \quad (35)$$

This equation is also valid for  $n = 0$ , viz. it is compatible with  $\lambda_0 \equiv 0$ .

Looking at the R.H.S. of Eq.(35), it turns out that we further need the equations for  $(l\ m\ n)$ ,  $\dot{K}_l$  and  $\langle \epsilon_{ab} \rangle_n$ . We investigate them one by one below.

### 3.2 Evolution equation for $(l \ m \ n)$

Applying Eq.(12) along with Eq.(9) to  $\langle f_l \rangle_{mn} = (l \ m \ n)$ , we get

$$(l \ m \ n) = \sum_{l'} \{ (l \ m \ l') \mu_{l'n} + (m \ n \ l') \mu_{l'l} + (n \ l \ l') \mu_{l'm} \} + \sum_{l'} (l \ m \ n \ l') K_{l'} . \quad (36)$$

Here, from Eq.(10) with Eq.(30) ( $N = 1$ ,  $N_a = 0$ ),  $\mu_{mn}$  is given by

$$\mu_{mn} = \begin{cases} \frac{D+1}{2(D-1)} \frac{1}{\lambda_m - \lambda_n} \sum_l \left( \lambda_n - \frac{D-3}{D+1} (\lambda_m - \lambda_l) \right) K_l(l \ m \ n) + \frac{2\sqrt{\lambda_m \lambda_n}}{\lambda_m - \lambda_n} \langle \epsilon_{ab} \rangle_{mn} & \text{for } m \neq n \\ -\frac{1}{2} \sum_l K_l(l \ m \ m) & \text{for } m = n \end{cases} . \quad (37)$$

### 3.3 Evolution equation for $K_l$

Now, since  $K_l = (K, f_l)$ , we get  $\dot{K}_l = (\dot{K}, f_l) + (K, \dot{f}_l) + (K, f_l K)$ , resulting in

$$\dot{K}_l = (\dot{K})_l + \sum_{l'} K_{l'} \mu_{l'l} + \sum_{l'l''} K_{l'} K_{l''} (l' \ l'' \ l) , \quad (38)$$

where Eq.(11) has been applied. To get the detailed expression for the first term  $(\dot{K})_l$ , we recall basic formulas of the canonical Einstein equations. We note

$$\dot{K} = -\frac{1}{2\alpha} \frac{D-1}{D-2} p - \frac{D-3}{2(D-2)} (\mathbf{R} + K^2 + \frac{D-1}{D-3} K \cdot K) + \frac{D-1}{D-2} \Lambda , \quad (39)$$

along with the constraints

$$\begin{aligned} \mathbf{R} - K \cdot K + K^2 - \frac{1}{\alpha} \rho - 2\Lambda &= 0 , \\ D^b K_{ab} - D_a K + \frac{1}{2\alpha} J_a &= 0 , \end{aligned} \quad (40)$$

where  $\alpha := \frac{c^3}{16\pi G}$  and  $\Lambda$  is the cosmological constant. Here we define the quantities  $\rho$ ,  $p$ ,  $J_a$  and  $S_{ab}$  in connection with the above equations: With the help of the normal unit vector  $n^\alpha$  of the spatial section, the energy-momentum tensor of matter  $T^{\alpha\beta}$  can be decomposed into three components:  $T^{\alpha\beta} = T_{\perp\perp}^{\alpha\beta} + T_{\parallel\perp}^{\alpha\beta} + T_{\parallel\parallel}^{\alpha\beta}$ , where each term has the following form,

$$T_{\perp\perp}^{\alpha\beta} = \rho n^\alpha n^\beta , \quad T_{\parallel\perp}^{\alpha\beta} = J^\alpha n^\beta + J^\beta n^\alpha , \quad T_{\parallel\parallel}^{\alpha\beta} = S^{\alpha\beta} .$$

(The suffix  $\perp$  implies “perpendicular to the space” and  $\parallel$  implies “along the space”.) Here  $J^\alpha$  and  $S^{\alpha\beta}$  are spatial quantities and they are uniquely identified to their spatial counterparts,  $J^a$  and  $S^{ab}$ , respectively. Then  $\rho$ ,  $J_a$  and  $S_{ab}$  are interpreted as the energy density, the momentum density and the stress tensor of

matter, respectively, and  $p := \frac{1}{D-1} S_a^a$  defines the pressure of matter. (The vector and tensor indices of spatial quantities in this context are lowered and raised by, respectively, the spatial metric  $h_{ab}$  and its inverse  $h^{ab}$ .)

Now, Eq.(39) can be modified by means of the first constraint equation (Hamiltonian constraint) in Eq.(40). In particular, the following two forms would be useful for our purposes. First, by eliminating  $\mathbf{R}$  from Eq.(39), we get

$$\dot{K} = -\frac{1}{2\alpha} \frac{D-3}{D-2} \left( \rho + \frac{D-1}{D-3} p \right) - K \cdot K + \frac{2}{D-2} \Lambda . \quad (41)$$

Second, by eliminating the term  $K \cdot K$  from Eq.(39), we get

$$\dot{K} = \frac{1}{2\alpha} \frac{D-1}{D-2} (\rho - p) - K^2 - \mathbf{R} + \frac{2(D-1)}{D-2} \Lambda . \quad (42)$$

Thus, we obtain the equation for  $(\dot{K})_l$  based on Eq.(41),

$$(\dot{K})_l = -\frac{1}{2\alpha} \frac{D-3}{D-2} \left( \rho_l + \frac{D-1}{D-3} p_l \right) - \frac{1}{D-1} \sum_{l'l''} K_{l'} K_{l''} (l' l'' l) + \frac{2\Lambda\sqrt{V}}{D-2} \delta_{l0} - (\epsilon \cdot \epsilon)_l . \quad (43)$$

In the same manner, based on Eq.(42), we get

$$(\dot{K})_l = \frac{1}{2\alpha} \frac{(D-1)}{(D-2)} (\rho_l - p_l) - \sum_{l'l''} K_{l'} K_{l''} (l' l'' l) + \frac{2(D-1)\Lambda\sqrt{V}}{D-2} \delta_{l0} - \mathbf{R}_l . \quad (44)$$

We also note that, from Eqs.(18), (32) and (A2),

$$\frac{\dot{V}}{V} = K_0 / \sqrt{V} = K_{\text{av}} . \quad (45)$$

The first constraint equation in Eq.(40) is translated into

$$\mathbf{R}_l + \frac{D-2}{D-1} \sum_{l'l''} K_{l'} K_{l''} (l' l'' l) - \frac{1}{\alpha} \rho_l - 2\Lambda\sqrt{V} \delta_{l0} - (\epsilon \cdot \epsilon)_l = 0 . \quad (46)$$

We also note that, taking the inner-product with  $\frac{1}{\lambda_l} D_a f_l$  ( $l \neq 0$ ), the Bianchi identity  $D^b \mathbf{R}_{ab} - \frac{1}{2} D_a \mathbf{R} = 0$  turns to

$$\mathbf{R}_l + \frac{2(D-1)}{(D-3)} \frac{1}{\lambda_l} (D^a D^b r_{ab})_l = 0 \quad (l \neq 0) . \quad (47)$$

Now, taking the inner-product with  $\frac{1}{\lambda_l} D_a f_l$  ( $l \neq 0$ ), the second constraint equation in Eq.(40) (momentum constraint) turns to

$$K_l + \frac{D-1}{D-2} \frac{1}{\lambda_l} \left\{ (D^a D^b \epsilon_{ab})_l + \frac{1}{2\alpha} (\vec{D} \cdot \vec{J})_l \right\} = 0 \quad (l \neq 0) . \quad (48)$$

Finally, noting that

$$\dot{\mathbf{R}} = -\mathbf{R}^{ab}\dot{h}_{ab} + D^a \left( D^b \dot{h}_{ab} - D_a(h^{cd}\dot{h}_{cd}) \right) ,$$

we get

$$\dot{\mathbf{R}}_l = \frac{D-3}{D-1} \sum_{l'l''} K_{l'} \mathbf{R}_{l''} (l' l'' l) + \sum_{l'} \mathbf{R}_{l'} \mu_{l'l} - \frac{1}{\alpha} (\vec{D} \cdot \vec{J})_l - 2(\epsilon \cdot r)_l , \quad (49)$$

where we used the momentum constraint Eq.(48) to reach the final form.

### 3.4 Evolution equation for $\langle \epsilon_{ab} \rangle_{l,mn}$ and $\langle r_{ab} \rangle_{l,mn}$

First, we derive the evolution equations for  $\epsilon_{ab}$  and  $r_{ab}$  ( $N = 1$ ,  $N_a = 0$ ),

$$\dot{\epsilon}_{ab} = \frac{1}{2\alpha} (S_{ab} - p h_{ab}) - \frac{D-3}{D-1} K \epsilon_{ab} - r_{ab} + 2(\epsilon \cdot \epsilon)_{ab} , \quad (50)$$

$$\begin{aligned} \dot{r}_{ab} = & -\frac{1}{D-1} \{ \Delta K h_{ab} + (D-3) D_a D_b K \} + \frac{1}{\alpha} \frac{1}{D-1} \vec{D} \cdot \vec{J} h_{ab} \\ & - \Delta \epsilon_{ab} - \frac{2}{D-1} \mathbf{R} \epsilon_{ab} + 2 D^c D_{(a} \epsilon_{b)c} + \frac{2}{D-1} \epsilon \cdot r h_{ab} . \end{aligned} \quad (51)$$

From Eq.(50) along with Eqs.(9) and (14), we get

$$\begin{aligned} \langle \dot{\epsilon}_{ab} \rangle_{l,mn} = & \frac{1}{2\alpha} \langle (S_{ab} - p h_{ab}) \rangle_{l,mn} - \left( \ln \sqrt{\lambda_m \lambda_n} \right) \langle \epsilon_{ab} \rangle_{l,mn} - \frac{2}{D-1} \sum_{l'l''} K_{l'} (l' l l'') \langle \epsilon_{ab} \rangle_{l'',mn} \\ & + \sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle \epsilon_{ab} \rangle_{l,mk} \mu_{kn} + \sum_k \sqrt{\frac{\lambda_k}{\lambda_m}} \langle \epsilon_{ab} \rangle_{l,nk} \mu_{km} \\ & + \sum_k \langle \epsilon_{ab} \rangle_{k,mn} \mu_{kl} - \langle r_{ab} \rangle_{l,mn} - 2 \langle (\epsilon \cdot \epsilon)_{ab} \rangle_{l,mn} . \end{aligned} \quad (52)$$

In the same manner, we get from Eq.(51),

$$\begin{aligned} \langle \dot{r}_{ab} \rangle_{l,mn} = & \langle \dot{r}_{ab} \rangle_{l,mn} - \left( \ln \sqrt{\lambda_m \lambda_n} \right) \langle r_{ab} \rangle_{l,mn} - \frac{5-D}{D-1} \sum_{l'l''} K_{l'} (l' l l'') \langle r_{ab} \rangle_{l'',mn} \\ & + \sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle r_{ab} \rangle_{l,mk} \mu_{kn} + \sum_k \sqrt{\frac{\lambda_k}{\lambda_m}} \langle r_{ab} \rangle_{l,nk} \mu_{km} \\ & + \sum_k \langle r_{ab} \rangle_{k,mn} \mu_{kl} - 4 \langle \epsilon^c_{(a} r_{b)c} \rangle_{l,mn} , \end{aligned} \quad (53)$$

where

$$\langle \dot{r}_{ab} \rangle_{l,mn} = \frac{1}{D-1} \sum_{l'l''} \lambda_{l'} K_{l'} (l' l l'') \langle h_{ab} \rangle_{l'',mn} - \frac{D-3}{D-1} \sum_{l'} K_{l'} \langle D_a D_b f_{l'} \rangle_{l,mn}$$

$$\begin{aligned}
& + \frac{1}{\alpha} \frac{1}{D-1} \sum_{l'l''} (\vec{D} \cdot \vec{J})_{l'} (l' l l'') \langle h_{ab} \rangle_{l'',mn} - \frac{2}{D-1} \sum_{l'l''} \mathbf{R}_{l'} (l' l l'') \langle \epsilon_{ab} \rangle_{l'',mn} \\
& - \langle \Delta \epsilon_{ab} \rangle_{l,mn} + 2 \langle D^c D_{(a} \epsilon_{b)c} \rangle_{l,mn} \\
& + \frac{2}{D-1} \sum_{l'l''} (\epsilon \cdot r)_{l'} (l' l l'') \langle h_{ab} \rangle_{l'',mn} .
\end{aligned} \tag{54}$$

For convenience, we also present the formulas Eq.(52) and Eq.(53) especially for  $l = 0$ :

$$\begin{aligned}
\langle \epsilon_{ab} \rangle_{mn} &= \frac{1}{2\alpha} \langle (S_{ab} - p h_{ab}) \rangle_{mn} - \left( \ln \sqrt{\lambda_m \lambda_n} \right) \langle \epsilon_{ab} \rangle_{mn} - \frac{2}{D-1} \sum_k K_k \langle \epsilon_{ab} \rangle_{k,mn} \\
& + \sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle \epsilon_{ab} \rangle_{mk} \mu_{kn} + \sum_k \sqrt{\frac{\lambda_k}{\lambda_m}} \langle \epsilon_{ab} \rangle_{nk} \mu_{km} - \langle r_{ab} \rangle_{mn} \\
& - 2 \langle (\epsilon \cdot \epsilon)_{ab} \rangle_{mn} .
\end{aligned} \tag{55}$$

$$\begin{aligned}
\langle r_{ab} \rangle_{mn} &= \langle \dot{r}_{ab} \rangle_{mn} - \left( \ln \sqrt{\lambda_m \lambda_n} \right) \langle r_{ab} \rangle_{mn} - \frac{5-D}{D-1} \sum_k K_k \langle r_{ab} \rangle_{k,mn} \\
& + \sum_k \sqrt{\frac{\lambda_k}{\lambda_n}} \langle r_{ab} \rangle_{mk} \mu_{kn} + \sum_k \sqrt{\frac{\lambda_k}{\lambda_m}} \langle r_{ab} \rangle_{nk} \mu_{km} \\
& - 4 \langle \epsilon^c_{(a} r_{b)c} \rangle_{mn} ,
\end{aligned} \tag{56}$$

where

$$\begin{aligned}
\langle \dot{r}_{ab} \rangle_{mn} &= \frac{1}{D-1} \sum_k \lambda_k K_k \langle h_{ab} \rangle_{k,mn} - \frac{D-3}{D-1} \sum_k K_k \langle D_a D_b f_k \rangle_{mn} \\
& + \frac{1}{\alpha} \frac{1}{D-1} \sum_k (\vec{D} \cdot \vec{J})_k \langle h_{ab} \rangle_{k,mn} - \frac{2}{D-1} \sum_k \mathbf{R}_k \langle \epsilon_{ab} \rangle_{k,mn} \\
& - \langle \Delta \epsilon_{ab} \rangle_{mn} + 2 \langle D^c D_{(a} \epsilon_{b)c} \rangle_{mn} + \frac{2}{D-1} \sum_k (\epsilon \cdot r)_k \langle h_{ab} \rangle_{k,mn} .
\end{aligned} \tag{57}$$

Eqs. (35)-(38), (43) (or (44)), (45)-(49), and (52)-(54) are the fundamental equations for the investigation of the spectral evolution of the Universe.

They form hierarchy equations and we can continue to get equations for higher hierarchies. This hierarchical property is a reasonable consequence since we are looking at the global quantities, and not the local ones. In practical applications, thus, we need to make a suitable truncation. Typically, we get equations of higher order in  $\epsilon_{ab}$  and  $r_{ab}$  when we continue to go into further hierarchies. We can often regard  $\epsilon_{ab}$ ,  $r_{ab}$ , and their spatial derivatives are small in cosmological applications. In such cases, the truncation becomes a reasonable approximation procedure.



## 4 Basic applications of the spectral equations

### 4.1 The Friedman-Robertson-Walker Universe

As a basic example, let us consider a closed Friedmann-Robertson-Walker Universe.

We set  $\epsilon_{ab} = 0$  and  $r_{ab} = 0$ . We also set  $\vec{J} = 0$ . From Eqs. (47) and (48), we get  $\mathbf{R}_l = 0$  and  $K_l = 0$  ( $l \neq 0$ ). Then, we get  $\rho_l = 0$  ( $l \neq 0$ ) from Eq.(46), noting that  $(0 \ 0 \ l) = \frac{1}{\sqrt{V}}\delta_{0l}$ . Eq.(38) along with Eq.(43) imply that  $p_l = 0$  ( $l \neq 0$ ), noting that  $\mu_{mn} = -\frac{1}{2}\delta_{mn}K_0/\sqrt{V}$  in the present case (Eq.(37)).

Now, Eq.(35) with Eq.(45) yield

$$\dot{\lambda}_n = -\frac{2}{D-1}\lambda_n \frac{\dot{V}}{V} ,$$

thus,

$$\lambda_n(t) = \left( \frac{V(0)}{V(t)} \right)^{\frac{2}{D-1}} \lambda_n(0) .$$

It is a simple scaling behavior expected from the dimensionality of  $\lambda_n$ .

Noting that  $\mu_{l0} = -\frac{1}{2}\delta_{l0}K_0/\sqrt{V}$  (Eq.(37)), we get from Eq.(49) for  $l = 0$  with Eq.(45),

$$\dot{\mathbf{R}}_0 = -\frac{5-D}{2(D-1)} \frac{\dot{V}}{V} \mathbf{R}_0 .$$

Thus,

$$\mathbf{R}_0(t) = \left( \frac{V(0)}{V(t)} \right)^{\frac{5-D}{2(D-1)}} \mathbf{R}_0(0) ,$$

or, noting Eq.(32),

$$\mathbf{R}_{\text{av}}(t) = \left( \frac{V(0)}{V(t)} \right)^{\frac{2}{D-1}} \mathbf{R}_{\text{av}}(0) .$$

It is also a simple scaling behavior expected from the dimensionality of  $\mathbf{R}$ .

On the other hand, from Eq.(46) for  $l = 0$  with Eq.(45), we get

$$\frac{D-2}{D-1} \left( \frac{\dot{V}}{V} \right)^2 + \mathbf{R}_{\text{av}} - \frac{1}{\alpha} \rho_{\text{av}} - 2\Lambda = 0 ,$$

or in a more familiar form,

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \frac{1}{(D-1)(D-2)} \frac{1}{\alpha} \rho_{\text{av}} - \frac{2}{(D-1)(D-2)} \Lambda = 0 ,$$

where we introduced the scale factor  $a$  as  $V \propto a^{D-1}$  and the curvature index  $k := \mathbf{R}_{\text{av}}(0)a(0)^2$ .

It can be solved once the matter content is specified.

## 4.2 A universal formula for the spectral distance between two Universes that are geometrically close to each other

The spectral evolution equations developed in the previous sections are of essential significance for the study of spacetime physics along the line of the spectral representation of geometrical structures.

We have the spectral distance  $d_N(\mathcal{G}, \mathcal{G}')$  as a measure of closeness between two spatial geometries  $\mathcal{G}$  and  $\mathcal{G}'$  [1]. It measures the difference between  $\mathcal{G}$  and  $\mathcal{G}'$  as the difference of “sounds” of them, i.e. the difference of the spectra.

Then we are given a basic arena for spacetime physics, i.e. the “space of all spaces”  $\mathcal{S}_N$ ; it is basically a space of all compact Riemannian geometries equipped with  $d_N(\mathcal{G}, \mathcal{G}')$  [4].

Since we now have the spectral evolution equations developed in the previous sections, we can investigate the time evolution of  $d_N(\mathcal{G}, \mathcal{G}')$ . The fundamental importance of such an investigation becomes clear when we take  $\mathcal{G}$  as the real Universe and  $\mathcal{G}'$  as a model Universe which is expected to be “close” to  $\mathcal{G}$  [6, 7]. There is no guarantee that the model Universe remains a good model for the real Universe in the future also [8, 9]. Stated in terms of the spectral distance, there is no guarantee that  $d_N(\mathcal{G}, \mathcal{G}')$  remains small in the future. The detailed investigations along this line would be done separately. Here we only look at some basic features of the evolution of  $d_N(\mathcal{G}, \mathcal{G}')$  when  $\mathcal{G}$  and  $\mathcal{G}'$  are very close initially in  $\mathcal{S}_N$ .

In this subsection, we derive a universal formula for  $d_N(\mathcal{G}, \mathcal{G}')$  when  $\mathcal{G}$  and  $\mathcal{G}'$  are geometrically close in  $\mathcal{S}_N$ , which is valid independently of the detailed form of  $d_N(\mathcal{G}, \mathcal{G}')$  nor the gravity theory. In the next subsection, we discuss its time evolution.

First, it is appropriate to recall basic settings of the spectral representation [1].

We set the eigenvalue problem on each manifold  $\mathcal{G}$  and  $\mathcal{G}'$ ,

$$\Delta f = -\lambda f \quad ,$$

then the set of eigenvalues (numbered in increasing order) is obtained;  $\{\lambda_m\}_{m=0}^{\infty}$  for  $\mathcal{G}$  and  $\{\lambda'_n\}_{n=0}^{\infty}$  for  $\mathcal{G}'$ .

Now the spectral distance between  $\mathcal{G}$  and  $\mathcal{G}'$  is defined as [4, 7]

$$d_N(\mathcal{G}, \mathcal{G}') = \sum_{n=1}^N \mathcal{F}\left(\frac{\lambda'_n}{\lambda_n}\right) \quad , \quad (58)$$

where  $\mathcal{F}(x)$  ( $x > 0$ ) is a smooth function which satisfies  $\mathcal{F} \geq 0$ ,  $\mathcal{F}(1/x) = \mathcal{F}(x)$ ,  $\mathcal{F}(y) > \mathcal{F}(x)$  if  $y > x \geq 1$  and  $\mathcal{F}(1) = 0$ . Then, it follows that  $\mathcal{F}''(1) \geq 0$ . However, in order to let  $d_N(\mathcal{G}, \mathcal{G}')$  detect a fine difference between  $\mathcal{G}$  and  $\mathcal{G}'$  when they are very close to each other in  $\mathcal{S}_N$ , we further postulate that  $\mathcal{F}''(1) > 0$ . (See Eq. (62).)

In particular, it is convenient to choose as  $\mathcal{F}$ ,  $\mathcal{F}_1(x) = \frac{1}{2} \ln \frac{1}{2}(\sqrt{x} + 1/\sqrt{x})$ . Then we get [1]

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{n=1}^N \ln \frac{1}{2} \left( \sqrt{\frac{\lambda'_n}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda'_n}} \right) . \quad (59)$$

It turns out that  $d_N(\mathcal{G}, \mathcal{G}')$  does not satisfy the triangle inequality. However, it does not cause a serious problem. Let  $Riem$  be the space of all  $(D-1)$ -dimensional, compact Riemannian geometries without boundaries. Then it is proved that the space  $(Riem, d_N)/\sim$ , where  $d_N$  is given by Eq.(59), is a metrizable space [4]. (Here  $\sim$  indicates identification of isospectral manifolds [5].) It justifies to regard  $d_N(\mathcal{G}, \mathcal{G}')$  as a distance, provided that we are careful when the triangle inequality matters in the argument.

The above property is shown in the following way. Since the breakdown of the triangle inequality turns out to be a mild one in a certain sense [4], it is expected that one can make a slight modification of  $\mathcal{F}_1$  to recover the inequality. Indeed we can find  $\mathcal{F}_0(x) := \frac{1}{2} \ln \max(\sqrt{x}, 1/\sqrt{x})$  as a modification of  $\mathcal{F}_1$ . In this case, Eq.(58) becomes

$$\bar{d}_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{n=1}^N \ln \max \left( \sqrt{\frac{\lambda'_n}{\lambda_n}}, \sqrt{\frac{\lambda_n}{\lambda'_n}} \right) .$$

It is easy to show that  $\bar{d}_N(\mathcal{G}, \mathcal{G}')$  satisfies all of the axioms of distance, so that it is a distance. Now, one can show that  $(Riem, d_N)/\sim$  and  $(Riem, \bar{d}_N)/\sim$  are homeomorphic to each other. Thus,  $(Riem, d_N)/\sim$  is a metrizable space since  $(Riem, \bar{d}_N)/\sim$  is a metric space. Let  $\mathcal{S}_N$  be a completion of  $(Riem, d_N)/\sim$ . The definition for  $d_N$  has a more convenient form than the one for  $\bar{d}_N$  since the latter includes  $\max$  symbol. Thus, in the actual applications,  $d_N$  is more useful than  $\bar{d}_N$ . On the other hand, when we need to discuss mathematical properties of  $\mathcal{S}_N$  precisely,  $\bar{d}_N$  is appropriate.

Let us consider the situation that  $\mathcal{G}$  and  $\mathcal{G}'$  possess the same topological structure and that they are very close in  $\mathcal{S}_N$ . One can imagine that  $\mathcal{G}' = (\Sigma, h')$  represents the real Universe at some instant of time, and  $\mathcal{G} = (\Sigma, h)$  is a model Universe corresponding to  $\mathcal{G}' = (\Sigma, h')$ . We introduce the difference in the spatial metric

$$\gamma_{ab} := h'_{ab} - h_{ab} , \quad (60)$$

and we treat  $\gamma_{ab}$  as a small quantity.

We regard the model Universe as a reference point, based on which we evaluate several quantities. In particular, we make use of the time-slicings of the model and investigate the time evolutions with respect to them.

Now, from Eq.(22), we get

$$\delta \ln \lambda_n = \frac{\lambda'_n - \lambda_n}{\lambda_n} = -\langle \bar{\gamma}_{ab} \rangle_n - \frac{1}{2} \langle \gamma \rangle_n , \quad (61)$$

where  $\gamma := h^{ab}\gamma_{ab}$ .

Looking at Eq.(58), we see that

$$\mathcal{F}\left(\frac{\lambda'_n}{\lambda_n}\right) = \mathcal{F}\left(1 - \left(\langle\bar{\gamma}_{ab}\rangle_n + \frac{1}{2}\langle\gamma\rangle_n\right)\right) = \frac{1}{2}\mathcal{F}''(1)\left(\langle\bar{\gamma}_{ab}\rangle_n + \frac{1}{2}\langle\gamma\rangle_n\right)^2 + O(\varepsilon^3) \quad .$$

Here we note that  $\mathcal{F}(1) = \mathcal{F}'(1) = 0$ , and  $\varepsilon$  indicates a small quantity in the same order as  $\gamma$ .

Leaving only the leading term, we thus get

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2}\mathcal{F}''(1) \sum_{n=1}^N \left(\langle\bar{\gamma}_{ab}\rangle_n + \frac{1}{2}\langle\gamma\rangle_n\right)^2 \quad . \quad (62)$$

Here we note the postulation  $\mathcal{F}''(1) > 0$ .

It would be also helpful to view Eq.(62) as

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2}\mathcal{F}''(1)\vec{\gamma} \cdot \vec{\gamma} \quad , \quad (63)$$

where  $\vec{\gamma}$  is a vector in  $\mathbf{R}^N$  whose  $n$ -th component is  $\langle\bar{\gamma}_{ab}\rangle_n + \frac{1}{2}\langle\gamma\rangle_n$ , and a standard Euclidean inner-product is implied.

Hence, we get a universal result on  $d_N(\mathcal{G}, \mathcal{G}')$  when  $\mathcal{G}$  and  $\mathcal{G}'$  are very close in  $\mathcal{S}_N$ : *The leading behavior of the spectral distance  $d_N(\mathcal{G}, \mathcal{G}')$  is given by Eq.(62) (or Eq.(63)), irrespective of the detailed form of the spectral distance nor of the gravity theory.*

In the case of Eq.(59), we have chosen as  $\mathcal{F}$ ,  $\mathcal{F}_1(x) = \frac{1}{2} \ln \frac{1}{2}(\sqrt{x} + 1/\sqrt{x})$ , hence  $\mathcal{F}''(1) = \frac{1}{8}$ . Thus, we get

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{16} \sum_{n=1}^N \left(\langle\bar{\gamma}_{ab}\rangle_n + \frac{1}{2}\langle\gamma\rangle_n\right)^2 \quad , \quad (64)$$

or

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{16}\vec{\gamma} \cdot \vec{\gamma} \quad , \quad (65)$$

where  $\vec{\gamma}$  is the same vector as in Eq.(63).

### 4.3 Time evolution of a small geometrical discrepancy between the real and a model Universes

Here we investigate the time evolution of the spectral distance for the two ‘nearby’ Universes as described in the previous subsection. Taking the time derivative of the both sides of Eq.(59), we get

$$\dot{d}_N(\mathcal{G}, \mathcal{G}') = \frac{1}{4} \sum_{n=1}^N \frac{\frac{\lambda'_n}{\lambda_n} - 1}{\frac{\lambda'_n}{\lambda_n} + 1} \left( \ln \frac{\lambda'_n}{\lambda_n} \right) \quad . \quad (66)$$

Now Eq.(35) can be represented as

$$(\ln \lambda_n)' = -\frac{2}{D-1} \sum_l \left(1 + \frac{D-3}{4} \frac{\lambda_l}{\lambda_n}\right) K_l(l \ n \ n) - 2\langle \epsilon_{ab} \rangle_n .$$

In the cosmological problems, it is often useful to separate the term for  $l = 0$  from the terms for  $l \geq 1$  in the summation like in the above equation: This separation is useful when the Universe is described by a homogeneous geometry plus small perturbations. Noting Eq.(45), we thus get

$$(\ln \lambda_n)' = -2 \left( \frac{1}{D-1} \frac{\dot{V}}{V} + \frac{1}{D-1} \langle K - K_{\text{av}} \rangle_n - \frac{D-3}{4(D-1)} \frac{1}{\lambda_n} \langle \Delta K \rangle_n + \langle \epsilon_{ab} \rangle_n \right) . \quad (67)$$

Let us define

$$\begin{aligned} H : &= \frac{1}{D-1} \frac{\dot{V}}{V} , \\ \iota_n : &= \frac{1}{D-1} \sum_{l \geq 1} \left(1 + \frac{D-3}{4} \frac{\lambda_l}{\lambda_n}\right) K_l(l \ n \ n) \\ &= \frac{1}{D-1} \langle K - K_{\text{av}} \rangle_n - \frac{D-3}{4(D-1)} \frac{1}{\lambda_n} \langle \Delta K \rangle_n , \\ \alpha_n : &= \langle \epsilon_{ab} \rangle_n . \end{aligned} \quad (68)$$

Here  $H$  is an analogous quantity to the Hubble constant;  $\iota_n$  is attributed to the inhomogeneity while  $\alpha_n$  is to the anisotropy of the geometry. Then, the formula (67) can be represented more concisely as

$$(\ln \lambda_n)' = -2H_n , \quad (69)$$

where

$$H_n := H + \iota_n + \alpha_n . \quad (70)$$

The quantity  $H_n$  can be interpreted as the effective Hubble constant observed at the scale  $\lambda_n^{-1/2}$  since it determines the rate of change of  $\lambda_n$ . Thus, Eq.(70) describes the modification of the *effective* Hubble parameter *at the scale*  $\lambda_n^{-1/2}$  due to inhomogeneity and anisotropy of the Universe *at that scale*.

Leaving only the leading terms in Eq.(66) with the help of Eqs.(69) and (70), we thus get

$$\dot{d}_N(\mathcal{G}, \mathcal{G}') = \frac{1}{4} \vec{\gamma} \cdot \delta \vec{H} , \quad (71)$$

where  $\delta \vec{H}$  denotes a vector in  $\mathbf{R}^N$  whose  $n$ -th component is  $\delta H_n := H'_n - H_n$ .<sup>7</sup> On the other hand, from Eq.(69) with Eq.(61), we can derive  $\vec{\gamma} = 2\delta \vec{H}$ , which is compatible with Eqs.(65) and (71).

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<sup>7</sup> In this subsection,  $\delta Q$  denotes the difference in a quantity  $Q$  for two spaces  $\mathcal{G}$  and  $\mathcal{G}'$ , defined as  $Q$  for  $\mathcal{G}'$  (the second entry of  $d_N(\cdot, \cdot)$ ) minus  $Q$  for  $\mathcal{G}$  (the first entry of  $d_N(\cdot, \cdot)$ ). Wherever necessary, we employ the notation  $\delta\{\cdot\}$  (rather than  $\delta(\cdot)$ ) to avoid any confusion with the  $\delta$ -function.

From Eq.(71), we get

$$\ddot{d}_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \delta \vec{H} \cdot \delta \vec{H} + \frac{1}{4} \vec{\gamma} \cdot \delta \dot{\vec{H}} \quad . \quad (72)$$

Let us introduce  $q_n := -(1 + \frac{\dot{H}_n}{H_n^2})$ , which can be interpreted as the effective deceleration parameter at the scale  $\lambda_n^{-1/2}$ . Then,  $\dot{H}_n = -(1 + q_n)H_n^2$ , so that

$$\delta \dot{H}_n = -2(1 + q_n)H_n \delta H_n - H_n^2 \delta q_n \quad . \quad (73)$$

Since  $\delta \dot{H}_n$  appears in Eq.(72) only in the form of  $\vec{\gamma} \cdot \delta \dot{\vec{H}}$ , it suffices to estimate  $\delta \dot{H}_n$  by leaving only the leading terms in Eq.(73). First, using Eq.(70), we note that

$$q_n \simeq (1 - \frac{2}{H}(\iota_n + \alpha_n))q - \frac{2}{H}(\iota_n + \alpha_n) - \frac{1}{H^2}(\dot{\iota}_n + \dot{\alpha}_n) \quad ,$$

where  $q := -(1 + \frac{\dot{H}}{H^2})$ .

Next, it is also straightforward to get estimations

$$\begin{aligned} (1 + q_n)H_n \delta H_n &\simeq \left[ (1 + q)\{H - (\iota_n + \alpha_n)\} - \frac{1}{H}(\dot{\iota}_n + \dot{\alpha}_n) \right] \delta H + (1 + q)H \delta \{\iota_n + \alpha_n\} \quad , \\ H_n^2 \delta q_n &\simeq H^2 \delta q + 2\{(1 + q)(\iota_n + \alpha_n) + \frac{1}{H}(\dot{\iota}_n + \dot{\alpha}_n)\} \delta H \\ &\quad - 2(1 + q)H \delta \{\iota_n + \alpha_n\} - \delta \{\dot{\iota}_n + \dot{\alpha}_n\} \quad . \end{aligned}$$

Looking at Eq.(73), we thus estimate

$$\delta \dot{H}_n \simeq -\delta \{(1 + q)H^2\} + \delta \{\dot{\iota}_n + \dot{\alpha}_n\} \quad .$$

Finally, we reach the estimation

$$\ddot{d}_N(\mathcal{G}, \mathcal{G}') \simeq \frac{1}{2} \delta \vec{H} \cdot \delta \vec{H} - \frac{1}{4} \left( \sum_{n=1}^N \gamma_n \right) \delta \{(1 + q)H^2\} + \frac{1}{4} \vec{\gamma} \cdot \delta \{\vec{\iota} + \vec{\alpha}\} \quad , \quad (74)$$

where  $\gamma_n := \langle \vec{\gamma}_{ab} \rangle_n + \frac{1}{2} \langle \gamma \rangle_n$  is the  $n$ -th component of  $\vec{\gamma}$ , and  $\vec{\iota}$  and  $\vec{\alpha}$  are vectors whose  $n$ -th components are  $\iota_n$  and  $\alpha_n$ , respectively. We note once again the definition of  $\delta Q$ , the difference of  $Q$  in  $\mathcal{G}$  and  $\mathcal{G}'$ , in this subsection (see the footnote just after Eq.(71)). The difference in metric,  $\gamma_{ab}$ , is also defined in the same manner (Eq.(60)). Thus, Eq.(74) is symmetric in  $\mathcal{G}$  and  $\mathcal{G}'$ , as it should be.

To make a detailed study, we need to investigate  $\langle \epsilon_{ab} \rangle_n$  and  $\langle r_{ab} \rangle_n$  also. It is helpful to note that  $\mu_{mn}$  in Eq.(37) gets simplified in the present case as

$$\begin{aligned} \mu_{mn} &= -\frac{1}{2} \frac{\dot{V}}{V} \delta_{mn} + O(\varepsilon) \\ &= -\frac{D-1}{2} H \delta_{mn} + O(\varepsilon) \quad . \end{aligned} \quad (75)$$

Now, we set  $m = n$  in Eq.(55). First, let us omit the first and the last terms on the R.H.S. of Eq.(55). Next, we leave only the  $k = 0$  part in the third term, which gives  $-\frac{2}{D-1}\frac{\dot{V}}{V}\langle\epsilon_{ab}\rangle_n = -2H\langle\epsilon_{ab}\rangle_n$  with the help of Eq.(45). Next, with the help of Eq.(75), the fourth and the fifth terms on the R.H.S. in Eq.(55) are approximated together as  $-\frac{\dot{V}}{V}\langle\epsilon_{ab}\rangle_n = -(D-1)H\langle\epsilon_{ab}\rangle_n$ . Finally, with the help of Eqs.(69) and (70), the second term in Eq.(55) yields  $2H\langle\epsilon_{ab}\rangle_n$  within the order of magnitude discussed now. Thus, we estimate

$$\dot{\langle\epsilon_{ab}\rangle}_n \simeq -(D-1)H\langle\epsilon_{ab}\rangle_n - \langle r_{ab}\rangle_n \quad . \quad (76)$$

In a similar manner, we can estimate  $\dot{\langle r_{ab}\rangle}_n$  by Eqs.(56) and (57). We set  $m = n$  in these equations. Among all the terms in Eq.(57), it is a reasonable approximation to leave only the  $k = 0$  part of the fourth term on the R.H.S., which gives  $-\frac{2}{D-1}\frac{\mathbf{R}_0}{\sqrt{V}}\langle\epsilon_{ab}\rangle_n$ . (This approximation is especially effective when we discuss the large-scale behaviors (low-lying spectra) such as  $\lambda_n^{-1/2} \gg c_s\tau$ , where  $c_s$  is the sound velocity of the matter and  $\tau$  is the typical cosmological time-scale of interest.) In Eq.(56), we omit the last term on the R.H.S. The third term can be estimated as  $-\frac{5-D}{D-1}\frac{\dot{V}}{V}\langle r_{ab}\rangle_n = -(5-D)H\langle r_{ab}\rangle_n$ , while the fourth and the fifth terms together are estimated as  $-\frac{\dot{V}}{V}\langle r_{ab}\rangle_n = -(D-1)H\langle r_{ab}\rangle_n$ . Finally, the second term yields  $2H\langle r_{ab}\rangle_n$ . Thus, we can estimate as

$$\dot{\langle r_{ab}\rangle}_n \simeq -2H\langle r_{ab}\rangle_n - \frac{2}{D-1}\mathbf{R}_{\text{av}}\langle\epsilon_{ab}\rangle_n \quad . \quad (77)$$

Now, we can estimate  $\ddot{d}_N(\mathcal{G}, \mathcal{G}')$  further based on Eq.(74). First,  $\dot{\alpha}_n$  is given by Eq.(76),

$$\dot{\alpha}_n \simeq -(D-1)H\alpha_n - \langle r_{ab}\rangle_n \quad . \quad (78)$$

Second, regarding  $\dot{\iota}_n$ , we go back to the definition of  $\iota_n$  (Eq.(68)). With the help of Eqs.(36), (38), (43), (69) and (75), it is straightforward to get estimations

$$\begin{aligned} \left(\frac{\lambda_l}{\lambda_n}\right)^\cdot &= O(\varepsilon) \quad , \\ (l \ n \ n)^\cdot &= -\frac{D-1}{2}H(l \ n \ n) + O(\varepsilon) \quad , \\ \dot{K}_l &= -\frac{5-D}{2}HK_l - \frac{D-3}{2(D-2)}\frac{1}{\alpha}(\rho_l + \frac{D-1}{D-3}p_l) + O(\varepsilon^2) \quad . \end{aligned}$$

Thus we get

$$\dot{\iota}_n \simeq -2H\iota_n - \frac{D-3}{2(D-2)}\frac{1}{\alpha}\mathcal{M}_n \quad , \quad (79)$$

where

$$\mathcal{M}_n := \frac{1}{D-1}\{\langle\rho-\rho_{\text{av}}\rangle_n + \frac{D-1}{D-3}\langle p-p_{\text{av}}\rangle_n\} - \frac{D-3}{4(D-1)}\frac{1}{\lambda_n}\{\langle\Delta\rho\rangle_n + \frac{D-1}{D-3}\langle\Delta p\rangle_n\} \quad .$$

With the help of Eqs.(78) and (79), Eq.(74) can be expressed as

$$\begin{aligned} \ddot{d}_N(\mathcal{G}, \mathcal{G}') \simeq & \frac{1}{2} \delta \vec{H} \cdot \delta \vec{H} - \frac{1}{4} \left( \sum_{n=1}^N \gamma_n \right) \delta \{ (1+q) H^2 \} \\ & - \frac{1}{4} \vec{\gamma} \cdot \delta \left\{ 2H\vec{r} + (D-1)H\vec{\alpha} + \vec{c} + \frac{(D-3)}{2(D-2)} \frac{1}{\alpha} \vec{\mathcal{M}} \right\} \quad , \quad (80) \end{aligned}$$

where  $\vec{c}$  and  $\vec{\mathcal{M}}$  are vectors whose  $n$ -th components are, respectively,  $\langle r_{ab} \rangle_n$  and  $\mathcal{M}_n$ .

Eqs. (65), (71) and (80) give us an idea on the factors that influence the validity of a cosmological model.

First, a particular combination  $\gamma_n := \langle \vec{\gamma}_{ab} \rangle_n + \frac{1}{2} \langle \gamma \rangle_n$  determines the spectral distance  $d_N(\mathcal{G}, \mathcal{G}')$ . Second,  $\vec{\gamma} \cdot \delta \vec{H}$  determines the rate of change of  $d_N(\mathcal{G}, \mathcal{G}')$ ,  $\dot{d}_N(\mathcal{G}, \mathcal{G}')$ . Whether  $d_N(\mathcal{G}, \mathcal{G}')$  decreases or not is governed by the relative directions of  $\vec{\gamma}$  and  $\delta \vec{H}$  in  $\mathbf{R}^N$ : One of the criteria for a good cosmological model would be that it makes the quantity  $\vec{\gamma} \cdot \delta \vec{H}$  negative, or non-negative and small at least. Third, the acceleration,  $\ddot{d}_N(\mathcal{G}, \mathcal{G}')$  is determined by several factors: The difference in the effective Hubble parameter,  $\delta \vec{H}$ , has always a repulsive effect. Even though  $\vec{\gamma} = 0$  initially, so that  $d_N = \dot{d}_N = 0$  initially, the spectral distance increases if  $\delta \vec{H} \neq 0$ . It is required that the other terms in Eq.(80) as a whole should be negative or at least non-negative and small in order to get a good model.

Let  $\tau$  be the typical time-scale with which we want to discuss the evolution of the Universe. Then we can summarize the criteria for a good cosmological model as follows:

- (C1)  $d_N(\mathcal{G}, \mathcal{G}')$  is small.
- (C2)  $\tau \dot{d}_N(\mathcal{G}, \mathcal{G}')$  is negative, or at least, non-negative and small.
- (C3)  $\tau^2 \ddot{d}_N(\mathcal{G}, \mathcal{G}')$  is negative, or at least, non-negative and small.

## 5 Discussion

In this paper, we have derived the spectral evolution equations of the Universe. A set of these equations forms one of the essential elements of the general scheme of spectral representation [1]: Now we have a space of all spaces [4],  $\mathcal{S}_N$ , equipped with the spectral distance [1],  $d_N$ , and the evolution equations on  $\mathcal{S}_N$ .

The spectral evolution equations are expected to be useful for studying the time evolution of the global geometrical structures of the Universe, since the spectra are especially suitable for describing global properties of a space.

The significance of the spectral evolution equations becomes prominent in situations when we need to handle a set of spaces rather than just one space, e.g.



the comparison between the real Universe and its model, the relation between the underlying topological structures and its low energy behavior (scale-dependent topology [2, 3]), and so on.

As one of the important applications of the spectral evolution equations, we can now investigate a fundamental problem in cosmology: Whether cosmology is possible, viz. under what conditions and to what extent a model reflects the real spacetime faithfully (“The averaging/model-fitting problem in cosmology [8, 9]).

As the first step in this direction, we have investigated in §4 the spectral distance between two very close Universes and its time evolution. It is interesting that the spectral distance in this situation is universally determined by the quantity  $\gamma_n := \langle \overline{\gamma}_{ab} \rangle_n + \frac{1}{2} \langle \gamma \rangle_n$  at each scale  $n$ , irrespective of the detailed form of the spectral distance nor the gravity theory (Eq.(63)). Even though it is a special case when two geometries are very close to each other in  $\mathcal{S}_N$ , this result would still provide us with a basic understanding about what kind of geometrical discrepancies would matter in the context of the validity of a cosmological model for the real Universe. More systematic studies along this line are certainly required, which would be presented elsewhere [10].

Finally, we make some remarks on the spectral scheme in general. It is a different way of viewing geometrical structures from the standard way of describing them. In this scheme, the geometrical information on a space is represented by a collection of the whole of the spectral information measured by all available elliptic operators on the space [1, 6, 7]. In ordinary cosmological observations, we use a particular observational apparatus so that we naturally get only a portion of the whole geometrical information on the space, according to which apparatus (or mathematically, which elliptic operators corresponding to the apparatus) has been utilized. Thus, it can happen that the geometry of the space cannot be fully identified by using just a single apparatus (some particular elliptic operators). This consideration provides us with a physical interpretation of the isospectral manifolds. From this viewpoint, there is no surprise in the existence of the isospectral manifolds. The spectral scheme describes the scale- and apparatus-dependent geometry of a space in a natural manner. According to what this scheme suggests us, there is no absolute model for the real Universe, rather a good model for reality depends on the observational scale which we are interested in, and on the observational apparatus which we rely on. The smaller scale we pay attention to and the more variety of apparatus we utilize, the closer the model Universe constructed from the data approaches the real Universe.

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## APPENDIX

## A The zero mode

In our theory, the zero-mode  $f_0$  of the Laplacian  $\Delta$  is more important than in a usual mathematical context: It is directly related to the spatial volume (Eq.(A2) below), so that it is a dynamical object just as other modes. Hence it is essential to note the basic facts on the zero-mode here for the development of our theory.

Since  $\Delta f_0 = 0$ , it follows that  $0 = \int f_0 \Delta f_0 \sqrt{V} = - \int (\partial f_0)^2 \sqrt{V}$ , implying that  $\partial_a f_0 \equiv 0$ . Thus,

The zero-mode of  $\Delta$  is a constant function, hence there is no degeneracy. (A1)

Now, from Eq.(1) for  $m = n = 0$ , it follows that  $1 = (f_0, f_0) = f_0^2 V$  on account of (A1). Here  $V$  is the  $(D - 1)$ -volume of the space. Thus<sup>8</sup>

$$f_0 = 1/\sqrt{V} \quad . \quad (A2)$$

Hence  $\int f_0 \sqrt{V} = \sqrt{V}$ . From Eq.(1) for  $n = 0$  with Eq.(A2), we get

$$\int f_m \sqrt{V} = \sqrt{V} \delta_{m0} \quad . \quad (A3)$$

Eq.(A2) can be represented as  $f_0 = (\int \sqrt{V})^{-1/2}$ . Taking the variation of the both sides of this equation, we easily get  $\delta f_0 = -\frac{1}{4V^{3/2}} \int g \cdot \delta g \sqrt{V}$ . With the help of Eq.(A2), we thus get

$$\delta f_0 = -\frac{1}{4} \langle g \cdot \delta g \rangle_0 f_0 \quad . \quad (A4)$$

This formula is of the fundamental importance to develop the perturbation theory suitable for our purpose (see *Appendix B* and §§2.3).

On the other hand, taking the variation of Eq.(A2) directly, we obtain

$$\delta f_0 = -\frac{1}{2} V^{-3/2} \delta V \quad . \quad (A5)$$

Comparing Eqs.(A4) and (A5) to each other, we get Eq.(18) in §2,

$$\langle g \cdot \delta g \rangle_0 = 2 \frac{\delta V}{V} \quad .$$

## B Basic results of the perturbation theory

The perturbation theory is helpful to to analyze the dynamical evolution of the spectra  $\{\lambda_n\}_{n=0,1,2,\dots}$ . In the present case, however, we need to pay special attentions to the zero-mode as is explained in *Appendix A*. We here derive basic

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<sup>8</sup> Here we choose the positive square-root. There is no essential difference even when we choose the negative square-root instead. For instance, Eq.(18) in §2 remains same.

formulas of the perturbation theory, taking care of the zero-mode. (Up to Eq.(B7) below, we mostly follow the argument in Ref.[11].)

We consider an Hermitian operator<sup>9</sup>  $\Delta$  parameterized by  $\alpha$  which is analytic in  $\alpha$  in the neighborhood of  $\alpha = 0$ :

$$\begin{aligned}\Delta &= \Delta_0 + \alpha\Delta_1 + \alpha^2\Delta_2 + \cdots \\ &=: \Delta_0 + \alpha\delta_\alpha\Delta \quad .\end{aligned}\tag{B1}$$

The operator  $\Delta$  can be arbitrary although we choose it to be the Laplacian in the application.

Let  $\{(\lambda_n^{(\alpha)}, |n\rangle_\alpha)\}_{n=0,1,2,\dots}$  and  $\{(\lambda_n^{(0)}, |n^{(0)}\rangle)\}_{n=0,1,2,\dots}$  be the set of spectra and eigenvectors for  $\Delta$  and  $\Delta_0$ , respectively:

$$\begin{aligned}\Delta|n\rangle_\alpha &= -\lambda_n^{(\alpha)}|n\rangle_\alpha \quad , \\ \Delta_0|n^{(0)}\rangle &= -\lambda_n^{(0)}|n^{(0)}\rangle \quad .\end{aligned}\tag{B2}$$

We understand that  $\{|n^{(0)}\rangle\}_{n=0,1,2,\dots}$  is the normalized set of eigenvectors. On the other hand, we do not specify the normalization of  $\{|n\rangle_\alpha\}_{n=0,1,2,\dots}$  at this stage (see after Eq.(B7) below).

We assume that both  $\lambda_n^{(\alpha)}$  and  $|n\rangle_\alpha$  are analytic in  $\alpha$  in the neighborhood of  $\alpha = 0$ :

$$\begin{aligned}\lambda_n^{(\alpha)} &= \lambda_n^{(0)} + \alpha\lambda_n^{(1)} + \alpha^2\lambda_n^{(2)} + \cdots \\ &=: \lambda_n^{(0)} + \alpha\delta_\alpha\lambda_n \quad , \\ |n\rangle_\alpha &= |n^{(0)}\rangle + \alpha|n^{(1)}\rangle + \alpha^2|n^{(2)}\rangle + \cdots \quad .\end{aligned}\tag{B3}$$

We introduce a projection operator  $P_n$  which projects any vector to the sector perpendicular to  $|n^{(0)}\rangle$ :

$$P_n := 1 - |n^{(0)}\rangle\langle n^{(0)}| = \sum_{k \neq n} |k^{(0)}\rangle\langle k^{(0)}| \quad .\tag{B4}$$

With the help of Eqs.(B1) and (B3), the first equation in (B2) can be represented as

$$(\lambda_n^{(0)} + \Delta_0)|n\rangle_\alpha = -\alpha(\delta_\alpha\lambda_n + \delta_\alpha\Delta)|n\rangle_\alpha \quad .\tag{B5}$$

The L.H.S. vanishes when multiplied by  $\langle n^{(0)}|$ , indicating that the R.H.S. of Eq.(B5) is perpendicular to  $|n^{(0)}\rangle$ . Thus, Eq.(B5) can be represented as

$$(\lambda_n^{(0)} + \Delta_0)|n\rangle_\alpha = -\alpha P_n(\delta_\alpha\lambda_n + \delta_\alpha\Delta)|n\rangle_\alpha\tag{B6}$$

Due to the presence of  $P_n$ , which removes the zero-mode of the operator  $\lambda_n^{(0)} + \Delta_0$ , Eq.(B6) can be expressed as

$$|n\rangle_\alpha = C_n(\alpha)|n^{(0)}\rangle - \frac{\alpha}{\lambda_n^{(0)} + \Delta_0} P_n(\delta_\alpha\lambda_n + \delta_\alpha\Delta)|n\rangle_\alpha \quad .\tag{B7}$$

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<sup>9</sup> It can also be regarded as an Hermitian-operator-valued function of  $\alpha$ .

From here, we need to proceed in a different way from the standard perturbation theory in quantum mechanics. Noting that  $\{|n\rangle_\alpha\}_{n=0,1,2,\dots}$  forms an orthogonal set of bases due to Eq.(B2), we can interpret  $C_n(\alpha) = \langle n^{(0)}|n\rangle_\alpha$  as the factor characterizing the normalization of  $\{|n\rangle_\alpha\}_{n=0,1,2,\dots}$ . It should be analytic in  $\alpha$  around  $\alpha = 0$  with  $C(\alpha) \rightarrow 1$  as  $\alpha \rightarrow 0$ :

$$C_n(\alpha) = 1 + \alpha c_n^{(1)} + \alpha^2 c_n^{(2)} + \dots \quad . \quad (\text{B8})$$

In the standard perturbation theory in quantum mechanics, the inner-product of the states is not perturbed. Thus any normalized state  $|A\rangle$  and its perturbation  $\delta|A\rangle$  are always perpendicular to each other, so that usually we can set  $C_n(\alpha) \equiv 1$  [11]. In our case, on the other hand, the inner-product itself is perturbed through the integral-measure  $\sqrt{\cdot}$ , when we regard  $(f_m, f_n)$  in Eq.(1) as the inner-product of the “states”  $f_m$  and  $f_n$ . Thus we should tread  $C_n(\alpha)$  with care (see the argument after Eq.(8)).

Since  $\langle n^{(0)}|(\delta_\alpha \lambda_n + \delta_\alpha \Delta)|n\rangle_\alpha = 0$ , we get

$$C_n(\alpha) \delta_\alpha \lambda_n = -\langle n^{(0)}|\delta_\alpha \Delta|n\rangle_\alpha \quad . \quad (\text{B9})$$

Eq.(B9) indicates that the both sides should match as analytic functions around  $\alpha = 0$ . Hence, taking into account Eqs.(B1), (B3) and (B8), we get

$$\begin{aligned} \lambda_n^{(1)} &= -\langle n^{(0)}|\Delta_1|n^{(0)}\rangle \quad , \\ \lambda_n^{(2)} &= -\langle n^{(0)}|\Delta_1|n^{(1)}\rangle - \langle n^{(0)}|\Delta_2|n^{(0)}\rangle - c_n^{(1)}\lambda_n^{(1)} \quad , \\ &\dots \\ \lambda_n^{(l)} &= -\langle n^{(0)}|\Delta_1|n^{(l-1)}\rangle - \langle n^{(0)}|\Delta_2|n^{(l-2)}\rangle - \dots - \langle n^{(0)}|\Delta_l|n^{(0)}\rangle \\ &\quad - c_n^{(l-1)}\lambda_n^{(1)} - c_n^{(l-2)}\lambda_n^{(2)} - \dots - c_n^{(1)}\lambda_n^{(l-1)} \quad , \\ &\dots \quad . \end{aligned} \quad (\text{B10})$$

Now, the R.H.S. of the second equation in (B3) and the R.H.S. of Eq.(B7) should match as vector-valued analytic functions around  $\alpha = 0$ . Hence, taking into account Eq.(B8), we get

$$\begin{aligned} |n^{(1)}\rangle &= \sum_k |k^{(0)}\rangle \mu_{kn}^{(1)} \quad , \\ |n^{(2)}\rangle &= \sum_k |k^{(0)}\rangle \mu_{kn}^{(2)} \quad , \end{aligned} \quad (\text{B11})$$

where

$$\begin{aligned} \mu_{mn}^{(1)} &:= \begin{cases} \frac{\langle \Delta_1 \rangle_{mn}}{\lambda_m^{(0)} - \lambda_n^{(0)}} & \text{for } m \neq n \\ c_n^{(1)} & \text{for } m = n \end{cases} \quad . \\ \mu_{mn}^{(2)} &:= \begin{cases} \frac{1}{\lambda_m^{(0)} - \lambda_n^{(0)}} \left( \langle \Delta_2 \rangle_{mn} + \lambda_n^{(1)} \mu_{mn}^{(1)} + \sum_k \langle \Delta_1 \rangle_{mk} \mu_{kn}^{(1)} \right) & \text{for } m \neq n \\ c_n^{(2)} & \text{for } m = n \end{cases} \quad . \end{aligned} \quad (\text{B12})$$

We note that Eq.(B9) can be expressed as

$$\delta_\alpha \lambda_n = -\frac{\langle n^{(0)} | \delta_\alpha \Delta | n \rangle_\alpha}{\langle n^{(0)} | n \rangle_\alpha} . \quad (\text{B13})$$

Thus  $\delta_\alpha \lambda_n$  should be independent of the normalization factor  $C_n(\alpha)$ . For example, with the help of Eq.(B11), the formula for  $\lambda_n^{(2)}$  in Eq.(B10) turns out to be

$$\begin{aligned} \lambda_n^{(2)} &= -\sum_{k \neq n} (\lambda_n - \lambda_k) \mu_{nk}^{(1)} \mu_{kn}^{(1)} - \langle \Delta_2 \rangle_n \\ &= -\sum_{k \neq n} \frac{\langle \Delta_1 \rangle_{nk} \langle \Delta_1 \rangle_{kn}}{\lambda_k^{(0)} - \lambda_n^{(0)}} - \langle \Delta_2 \rangle_n , \end{aligned} \quad (\text{B14})$$

which is independent of  $c_n^{(l)}$  ( $l = 1, 2, \dots$ ).

Mostly  $\alpha$  is identified with a time-function  $t$  in this paper. When we interpret the variation  $\delta Q$  of any quantity  $Q$  to be a derivative of a corresponding function  $Q(\alpha)$  at  $\alpha = 0$ , only the terms of order  $O(\alpha)$  (quantities with superscript indices (0) and (1)) in the above formulas are important.

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